Factors affecting the stability of viscous axial flow in annuli with a rotating inner cylinder

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Hot-wire measurements are presented of the onset of instability in developed axial flow and in both developing and developed tangential flow caused by inner cylinder rotation in concentric annuli of radius ratio *N* of 0.909, 0.809 and 0.565 for axial-flow Reynolds numbers (*Re*) between 86 and 2000. Within assessed uncertainty intervals, the consistency of marginal stability measurements, at four azimuthal locations 90° apart, indicates insensitivity to small variations in gap width; the measurements also confirm the destabilisation of nearly-developed and developed tangential flow identified by Takeuchi and Jankowski[1] with the occurrence at increasing *Re* of three-dimensional initial disturbances of spiral-vortex form. Comparison with earlier measurements suggests that in particular annuli, destabilisation may be delayed to higher *Re* by high values of certain geometrical factors, including radius ratio and the resultant end-effects parameter. Stability may also be restored or improved at high *Re* by reversion to developing tangential flow in which the initial instability is not of spiral-vortex form and where, for given *N*, the critical Taylor number appears uniquely related to the dimensionless axial co-ordinate. Stability is then generally greatest at low *N*.

Key words: *vortices, flow stability, secondary flows*

Numerous processes in the chemical, mechanical and electrical engineering industries involve fluid flow through annuli with rotating inner cylinders and stationary outer cylinders: the cooling of rotating electrical machinery by rotor-mounted fans, the return flow of drilling mud from oil-drilling cores, operation of the rectifying section of rotary concentric-tube distillation columns, drying and paper-making machinery, and the lubrication of plain bearings are but a few examples. Reliable knowledge of the fluid motion is required by the designer of such equipment, including the ability to predict the conditions under which secondary flows will occur in the tangential boundary layer created by inner cylinder rotation, because such flows influence the torque and heat-transfer characteristics in relation to the transport properties of the working fluid. We therefore need to be able to determine the onset of rotational instability in terms of critical Taylor number Ta_c and the forms such instability can take over fairly wide ranges of working fluid, annulus radius ratio *N* and axial flow Reynolds number *Re*.

Following the classical work of Taylor[2] on Couette flow at zero *Re*, almost all the extensive theoretical and experimental research on the problem for non-zero *Re* has been carried out in the last quarter-century, with heavy concentration on the case of the fully-developed laminar tangential boundary layer. Less attention has been devoted to the equally, if not more, important practical case of the upstream region in which the developing tangential boundary layer, within which instability is initiated, is a function of the dimensionless axial co-ordinate:

$$l = 2z/[Re\,(R_2 - R_1)]$$

The axial distance *z*, from the point at which inner cylinder rotation begins, is usually preceded in experimental investigations by a stationary inner length L_{1s} to allow complete development of the axial flow.

For developed tangential flow, the earlier linear stability analyses of Chandrasekhar[3–5], Di Prima[6], Krueger and Di Prima[7], Datta[8] and Elliott[9], and more recently those of Hasoon and Martin[10] and Di Prima and Pridor[11], assume a steady finite-amplitude secondary motion consisting of toroidal vortices distributed periodically along the axis of the cylinders, as is the case for circular Couette flow. While the assumption of axi-symmetry may be justified for small *Re*, it leads to the conclusion that Ta_c increases monotonically with *Re*, as was found by Hasoon and Martin[10], who relaxed the earlier restriction to a narrow gap and considered *Re* as high as 2000.

The numerical predictions of Chung and Astill[12] for an arbitrary gap using a shooting method are in fact based on a non-axisymmetric disturbance, but their wavenumbers do not agree with those measured by Snyder[13,14]. Moreover, their treatment has been questioned by Di Prima and Pridor[11] and Takeuchi and Jankowski[1], who view as inconsistent and theoretically unjustified the assumption of Chung and Astill in their minimisation process that Ta_c

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increases monotonically with *Re* for all azimuthal wavenumbers. At sufficiently large *Re* monotonic stabilisation of circular Couette flow by an axial flow must be overtaken by viscous instability in the parallel shear flows inherent in spiral Poiseuille flow even without the centrifugal instability due to rotation, as is shown by Mott and Joseph[15] and supported by the measurements of Snyder[13,14]; for N of 0.95 and 0.96 toroidal vortices are replaced by non-axisymmetric three-dimensional spiral vortices when $Re > 30$. Takeuchi and Jankowski point out that then both an axial wavenumber and an integer azimuthal wavenumber must be involved in treating the stability problem. In their linearised analysis for N of 0.5 they find the stability limit as the minimum on the family of non-monotonic overlapping neutral stability curves generated by varying both wavenumbers over their proper ranges. When linked to form an envelope, the branches of these curves indicate that Ta_c ceases to increase with *Re* at $Re = 120$ and decreases as *Re* is further increased.

While in qualitative agreement, the measurements of Takeuchi and Jankowski for $N = 0.5$ yield up to 28% larger Ta_c than predicted (as shown in Fig 8); this they attribute to the vortex development length (that for a moving disturbance to reach an observable amplitude using visualisation techniques, rather than that for Ta_c to achieve independence of l) possibly extending at high *Re* beyond the hydrodynamic entrance length. Though their annulus design amply satisfies the predictions of Martin and Payne[16] for developed tangential flow, measured Ta_c might have been lower if L_{1r} had been sufficient for disturbances to reach threshold amplitude. If annulus length is influential, a non-linear analysis is needed to determine the conditions for successful observation of a vortex structure; these conditions should include the sensitivity of the apparent location of neutral stability to the method of observation in both axial and radial directions.

Destabilising trends at low *Re* are also found in the measurements of Cornish[17] for $N = 0.977$ and Sorour and Coney[18] for $N = 0.8$, suggesting that the destabilising influence of spiral vortices affects flows over a wide range of N. But in many other observations Ta_c continues to increase monotonically with *Re*, usually according to a power-law relation, to much larger *Re*; in Williamson's measurements[19] for $N = 0.9$ to $Re = 500$, and in those of Sorour and Coney[18] for $N = 0.955$ to $Re > 1000$ before the rate of increase of Ta_c even starts to diminish. This suggests the presence of other factors which offset the destabilising effect of increasing azimuthal wave number. Non-axisymmetric spiral vortices might also cause a measurable circumferential variation of Ta_c, although Gravas and Martin[20] believe that the variations of up to 93% which they observed in the high *Re* range of destabilisation for $N = 0.9$ were more likely to be the result of circumferential and axial variations in gap width. These are difficult to avoid when machining long cylinders of nearly equal diameter and Gravas and Martin recommended that further measurements be made by way of confirmation.

Sorour and Coney[18] have, like Takeuchi and Jankowski, considered the possible influence of L_{1r} on apparent stability, but indirectly by suggesting that the latter is linked to end effects and therefore to the annulus length/gap width ratio. They also pursued Snyder's analytical conclusion[13] that with axial flow, stability might be determined by the number of times a streamline encircled the rotating section of the annulus throughout its length L_{1r}. Early destabilisation could therefore be prevented by sufficiently high values of either the geometrical factor S', or the end effects parameter S, given respectively by:

$$S' = \frac{L_{1r}}{6\pi}\left[\frac{2}{R_2^2 - R_1^2}\right]^{1/2} = \frac{0.075 L_{1r}}{(1-N^2)^{1/2} R_2} \tag{1}$$

$$S = S'\frac{Ta^{1/2}}{Re} = \frac{L_{1r}}{6\pi Re}\left[\frac{2Ta}{R_2^2 - R_1^2}\right]^{1/2} = \frac{0.075 N\omega L_{1r}}{(1+N)\overline{W}} \tag{2}$$

A further stabilising factor, first suggested by Hasoon and Martin[10] in connection with the measurements of Kaye and Elgar[21] and Becker and Kaye[22] for $Re > 400$, is a reversion from developed to developing tangential flow if *Re* is large enough for l to fall below 0.2, the minimum predicted by Martin and Payne[16] for developed tangential flow. In developing flow Ta_c has been found experimentally by Martin and Payne[16], Payne and Martin[23] and Martin and Hasoon[24] to obey an inverse power-law relationship with l; thus Ta_c increases with *Re*. The greater stability of developing tangential flow derives from an

Notation

L_{1r}	Length of rotating section of inner cylinder
L_{1s}	Stationary approach length of inner cylinder
L_2	Overall length of stationary outer cylinder
l	Dimensionless axial distance from start of inner cylinder rotation, $2z/[Re\,(R_2 - R_1)]$
N	Annulus radius ratio, R_1/R_2
R_1	Outer radius of inner cylinder
R_2	Inner radius of outer stationary cylinder
Re	Axial flow Reynolds number, $2\overline{W}(R_2 - R_1)/\nu$
S	End effects parameter, $S' Ta^{1/2}/Re$
S'	Geometric factor, $(L_{1r}/6\pi)/[2/(R_2^2 - R_1^2)]^{1/2}$
Ta	Taylor number, $4\omega^2 R_1^2 (R_2 - R_1)^3/[\nu^2 (R_1 + R_2)]$
$\overline{W}$	Mean axial velocity of fluid
z	Axial distance downstream from start of inner cylinder rotation
δ_θ	Tangential boundary-layer displacement thickness
ν	Kinematic viscosity of fluid
ω	Angular velocity of rotating inner cylinder

Subscript

c	Critical value for onset of instability

initial disturbance which originates as oscillating ripples near the rotating surface; as visualised by Astill[25,26], the wave motion progresses across the growing tangential boundary layer finally to curl over to form trapezoidal vortices as the flow becomes fully established.

This paper first attempts to resolve uncertainties about the presence and detectability of non-axisymmetric spiral vortices as orientational variations of Ta_c for developed and developing tangential flow over an *Re* range from 86 to 2000, in annuli of radius ratio 0.909, 0.809 and 0.565. Hot-wire measurements are presented for the onset of instability at four circumferential locations 90° apart at a series of stations along the annulus where special precautions were taken to minimise variations in gap width, particularly circumferentially, arising from manufacturing and positional tolerances. The resulting percentage orientational variations in Ta_c are so much less than those recorded by Gravas and Martin[20] as to be within the limits of experimental uncertainty. Whatever its mode, the initial instability appears insensitive to small variations in gap width and non-axisymmetry could not be detected in terms of circumferential variation of Ta_c.

Our measurements also confirm that in developing tangential flow Ta_c always increases with *Re* through an inverse power-law relation with l, the form of relationship depending on N. As the flow approaches full tangential development, however, Ta_c becomes increasingly independent of *Re*, although the region of maximum Ta_c is usually at larger *Re* than 120, the value associated with spiral vortex flow by Takeuchi and Jankowski[1] for $N=0.5$. Further evaluation of our measurements in relation to the above discussion of factors affecting flow stability involves comparison with a number of predictions and 21 sets of earlier measurements. For developed tangential flow, neutral stability measurements fall into three groups, depending on the *Re* range in which Ta_c increases with *Re* and the maximum Ta_c. Spiral vortices clearly destabilise the flow, but a reversion to developing tangential flow has the opposite effect, as do sufficiently high values of S', S and N for other than very low *Re*; even one of these factors can increase stability above what might have been expected. On the other hand, an increase in N destabilises developing tangential flow and, as measured by Ta_c, the dimensionless flow development length with an initial instability increases above $l=0.2$ as *Re* decreases.

Experimental apparatus and procedure

In the experimental rig (Fig 1) which was fully described by Grosvenor[27], air blown from the atmosphere is metered by free-float rotameter en route to a conical chamber and thence through a multiplicity of plastic hose connections to circumferential inlets to a cylindrical stilling chamber. This houses and centralises the upstream part of the test annulus, the outer stationary cylinder of overall length L_2 being foreshortened so that air enters the test annulus by flow reversal, having previously passed through the annular gap between the stilling-chamber wall and the outer cylinder in the opposite direction. The stilling chamber is carried in two yoke-like supports constructed to allow vertical adjustments to the test annulus to achieve a horizontal setting.

The inner cylinder of the test annulus comprises two axial sections. The upstream stationary approach length L_{1s}, part of which lies within the stilling chamber, is sufficient according to the criteria of Sparrow and Lin[28] for 99% development of the axial velocity profile for *Re* up to 2000. The upstream end of this section is held rigidly by the end support plate of the stilling chamber while its downstream end locates the upstream end of the rotating section, of length L_{1r}, via a bearing-and-housing arrangement. The downstream end of the rotating section is carried by a ball bearing housed in a support arrangement which allows vertical and horizontal adjustment during positioning of the cylinders. The rotating section, which is driven by an electric motor with speed control through a linkage V-belt and pulleys, is fitted with end spigots assembled in the cylinder before final machining to ensure their concentricity with the outer surface of the cylinder. After passage through the annulus the air is discharged to atmosphere, its temperature being monitored at the rotameter exit, the

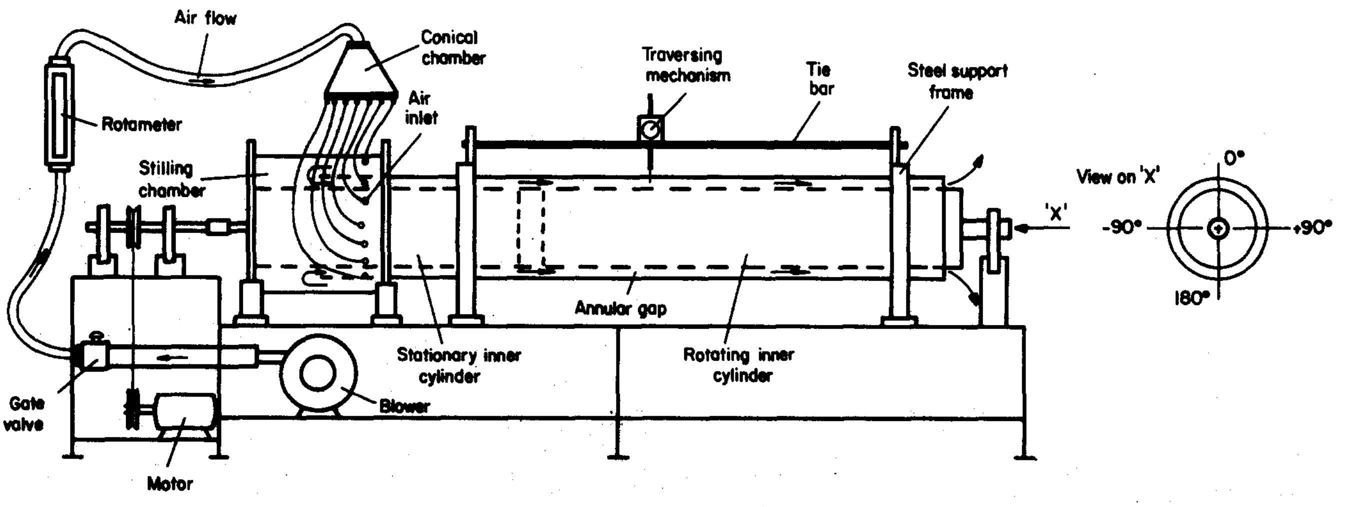

Fig 1 Arrangement of experimental rig

stilling chamber and the annulus exit. Dimensions and other details of each of the test annuli are given in Table 1.

The stationary outer cylinder, which is mounted in a supporting framework with adjustment points to allow the annulus to be assembled concentrically, is provided with four longitudinal rows of 3 mm diameter access holes (normally closed with nylon plugs) 90° apart for hot-wire anemometer studies. Thus a hot-wire probe can be inserted at sufficient axial locations to measure marginal instability around the annulus (at each location) in both developed and developing tangential flow. The supporting framework also carries the hot-wire traversing mechanism, on tie-bars which are removable and adjustable to the orientation to be studied. The traversing mechanism can be locked at the selected axial location so that the hot-wire sensor assembly, which incorporates a probe of 5 μm tungsten wire, enters a given access hole centrally and perpendicularly.

After initial assembly the annular gap width was measured by introducing into each access hole a specially-designed traversing probe which completed an electrical circuit when in contact with either annulus surface. This was used in conjunction with an optical measuring device to minimise variations in gap width under both stationary and rotating conditions by successive adjustments to the positioning of the cylinders forming the annulus. A similar electrical arrangement, consisting of a dummy probe in conjunction with a potentiometer, was used to calibrate the traversing mechanism in terms of radial movement of the hot-wire probe across the gap. Hot-wire probe readings were normally taken at six radial positions with the wire filament perpendicular to the axial air flow. The anemometer output voltage was amplified and fed to an oscilloscope, the rotational speed of the inner cylinder measured by tachogenerator being displayed on a digital dc voltmeter.

Following the establishment of steady-state conditions at the desired Re, the inner cylinder rotational speed is increased from zero to a value where ripples appear on the previously-steady oscilloscope trace for the location being monitored, with a corresponding jump in rms voltage. These ripples increase in intensity with further increase in rotational speed, their development being well illustrated by Payne and Martin[23]. The concept of the first discernible ripple as an indicator of marginal instability was first introduced by Astill[25,26] but is somewhat arbitrary in that detection depends partly on the sensitivity of the monitoring equipment. Therefore to obtain Ta_c as accurately as possible, the average of two measurements is taken, one when entering the unstable region from low rotational speeds, the other when leaving it from high rotational speeds. The difference between the two speeds is typically within 3% of the average value.

Such averages are obtained for each access hole at the six radial locations previously mentioned; the values of Ta_c, subsequently presented, result from extrapolating to the rotating surface a smooth curve through these averages, as reported by Hasoon and Martin[10] and illustrated by Gravas and Martin[20]. Because the critical Taylor number diminishes monotonically, and in some cases approaches uniformity, with decreasing radius, its lowest value, and that presented below, occurs at the rotating surface. The measurements thus confirm the findings of Gravas and Martin that instability does not originate within the flow; as will be seen below, within the limits of experimental uncertainty they also provide little evidence of sensitivity to non-axisymmetry of the initial instability or to small variations in gap width.

Table 1 Nominal dimensions and other details of test annuli

N	0.909	0.809	0.565
R_2, mm	69.85	69.85	100
R_1, mm	63.50	56.49	56.49
$R_2 - R_1$, mm	6.35	13.36	43.51
$R_2^2 - R_1^2$, mm^2	846.77	1687.90	6808.88
L_2, m	2.0	2.0	2.1
L_{1s}, mm	677	880	880
L_{1r}, m	1.37	1.18	1.25
S'	3.53	2.15	1.13

Experimental measurements and observations

In the annulus of $N = 0.909$ the overall mean gap width with inner cylinder rotation was 6.34 mm, based on 64 circumferential and axial measurements over a 1.3 m length; the maximum and minimum mean gap widths of 6.58 mm and 6.90 mm (giving a maximum deviation from the mean of 3.9%) occurred at opposite ends of a diameter at the same axial location ($z = 720$ mm). Fig 2 shows measurements of Ta_c

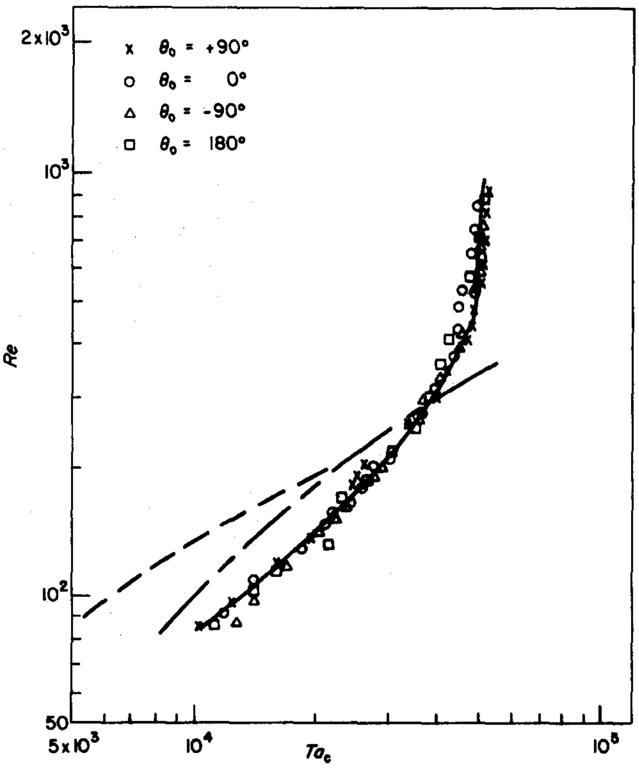

Fig 2 Marginal stability measurements at four azimuthal locations around the annulus of N = *0.909 at* z = *720 mm. - - -, prediction of Di Prima and Pridor*[11] *for* N = *0.9; — — —, prediction of Hasoon and Martin*[10] *for* N = *0.893*

for all four orientations at this station (of maximum gap-width variation) over the range $86 \leq Re \leq 919$. The latter value corresponds to a minimum l of 0.238; hence by the criterion of Martin and Payne[16] the flow is tangentially fully developed. The small scatter of data points, which are readily correlated by a single curve, is generally well within the experimental uncertainty interval of ±6.8%, as assessed by the procedure of Kline and McClintock[29], and there is no tendency for measurements for different azimuthal orientations to follow separate paths, as was observed by Gravas and Martin[20] for N of 0.9 and 0.81.

The departure of Ta_c from its monotonic increase with Re, which is attributed to the development of spiral vortex flow of increasing azimuthal wavenumber, occurs when $Re = 450$, a considerably higher value than that of 120 reported by Takeuchi and Jankowski[1] for $N = 0.5$. For $Re = 450$, $Ta_c \simeq 4.8 \times 10^4$ and, rather than decrease, it remains approximately constant with further increase in Re. The calculations of Di Prima and Pridor[11] for $N = 0.9$ shown in Fig 2 underpredict measurements of Ta_c by over 100% at $Re = 86$ and while this falls to about 25% at $Re = 200$ the convex curvature of the predictions is throughout at variance with the concavity of the measurements and precludes any tendency of Ta_c to independence of Re. Though in better agreement with measurements, this is equally true of the finite-difference predictions of Hasoon and Martin[10] for $N = 0.893$, also included in Fig 2, and results in both cases from the assumption of axisymmetric disturbances so convincingly challenged by Takeuchi and Jankowski for other than very low Re. The axisymmetry of measurements of Ta_c in Fig 2 demonstrates the insensitivity of the initial disturbance to small variations in gap width; also that the non-axisymmetry of spiral vortex flow is not reflected in orientational variations of Ta_c.

These conclusions are reinforced by similar unpresented observations for other axial locations. Fig 3, therefore, shows only mean measurements of Ta_c for z of 100, 150, 250, 720 and 1320 mm along the surface of the annulus. Measurements for 1320 mm coincide with those for $z = 720$ mm (discussed above) within 12% throughout the Re range covered, thus indicating the existence of developed tangential flow. At $z = 250$ mm, there is coincidence with measurements for $z \geq 720$ mm only when Re exceeds 450. As in Fig 2, Ta_c becomes independent of Re in this region provided that $z \geq 250$ mm. For lower z the tangential flow is still developing; thus for z of 150 mm and 100 mm, Ta_c shows no tendency to independence of Re in Fig 3. This accords with Astill's[25,26] observations that in developing tangential flow the initial instability is not of spiral vortex form.

The superposition of the curve for $l = 0.2$ in Fig 3 may be related to Fig 4 in which Ta is plotted against l with Re as parameter, thus covering both developing and developed tangential flow. For $Re = 100$, Ta_c achieves independence of l only when the latter exceeds 2, but at $Re = 800$ this falls to little more than 0.1. Martin and Payne's criterion of $l \geq 0.2$ for developed tangential flow (in the absence of instability) is therefore satisfactory only for $Re > 450$; at least for $N = 0.909$ the minimum l varies inversely with Re, though Astill[30] has queried the implication that the minimum values of l for developed tangential flow with and without an initial instability are necessarily the same. At larger l the trend in Fig 4 for constant Re curves to collapse reflects the region of independence of Ta_c in Fig 3. At low l, Ta_c is related to Re only through l, the tangent to the curves in Fig 4 being described by the equation:

$$Ta_c = \frac{10700}{l^{0.62}} \tag{3}$$

This and similar correlations for other N are discussed further below.

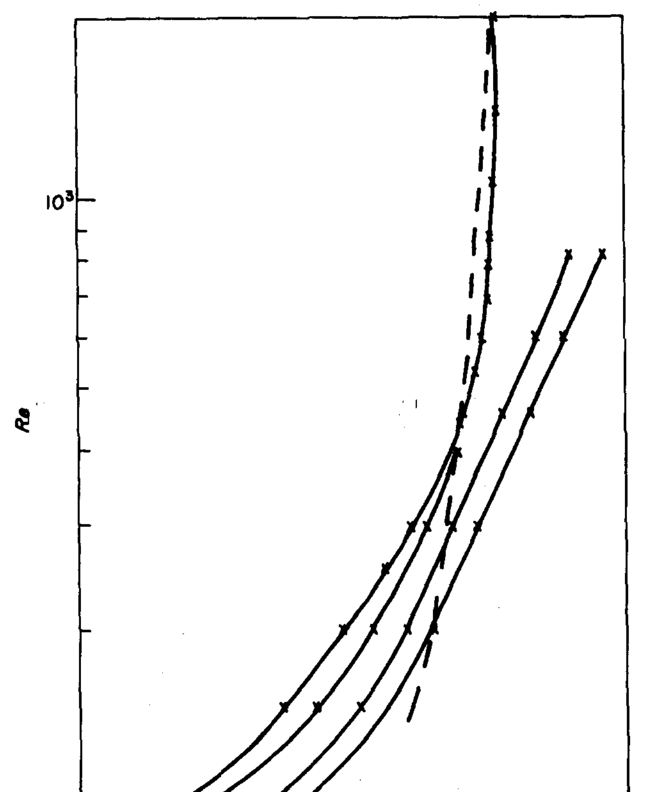

Fig 3 Mean marginal stability measurements in annulus of N = *0.909 for* z *(from right to left) of 100, 150, 250, 720 and 1320 mm; ---, locus of* $l = 0.2$

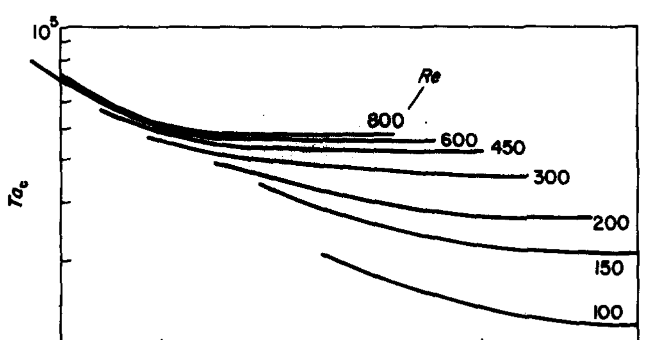

Fig 4 Mean marginal stability measurements as a function of the dimensionless axial co-ordinate l *for various* Re *in annulus of* N = *0.909*

For the annulus of $N = 0.809$ the overall mean gap width was 13.15 mm; the maximum and minimum mean gap widths of 13.76 mm and 12.59 mm gave a maximum deviation from the mean of 4.6%. Circumferential measurements at the largest z of 1180 mm typically showed a maximum variation in Ta_c of 6.9% over the range $92 \leq Re \leq 918$, again within the assessed uncertainty interval of ±7.3% for this annulus radius ratio. As before, Fig 5 therefore shows only mean measurements of Ta_c for z of 60, 110, 180, 280, 480 and 1180 mm. These portray a similar trend to independence of Re above about 700 but only in the downstream region approaching developed tangential flow, where $z > 180$ mm. Coincidence of measurements is confined to the region of $Re = 100$, and then only for z of 480 and 1180 mm, where $l \geq 0.73$, indicating that the locus $l = 0.2$ in Fig 5 is an inadequate criterion, and that elsewhere tangential flow is not developed. This is again borne out by the relation between Ta_c and l for various Re in Fig 6, which differs from Fig 4 in that Ta_c appears to be separately sensitive to Re even at low l. In this

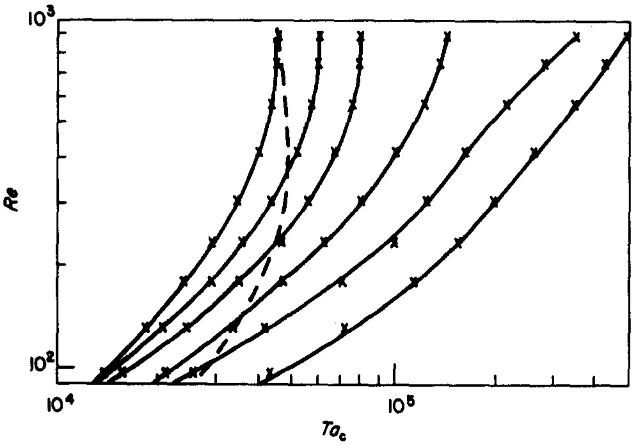

Fig 5 Mean marginal stability measurements in annulus of N = *0.809 for* z *(from right to left) of 60, 110, 180, 280, 480 and 1180 mm; – – –, locus of* l = *0.2*

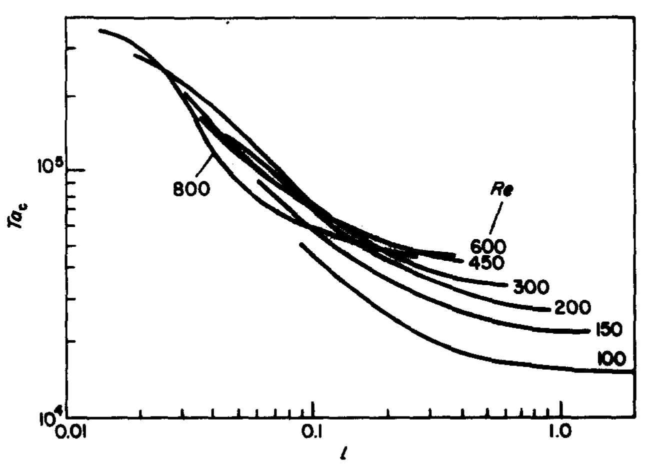

Fig 6 Mean marginal stability measurements as a function of the dimensionless axial co-ordinate l for various Re *in annulus of* N = *0.809*

region the slope of the curves is less uniform, the best fit being given by:

$$Ta_c = \frac{3223}{l^{1.16}} \tag{4}$$

In the annulus of $N = 0.565$ the overall mean gap width was 43.62 mm. The maximum and minimum mean gap widths were 46.27 mm and 41.09 mm, giving a maximum deviation from the mean of 6.1%. The low rotational speeds at which instability was initiated in this annulus made for considerable difficulty in minimising fluctuations of speed and its accurate measurement because of the small output voltages fed from the tachogenerator, which were sometimes within the sensitivity range of the recording voltmeter. The relatively large errors in measuring ω_c are reflected in the scatter of orientational Ta_c measurements in Fig 7 for $z = 1195$ mm, compared with an assessed uncertainty interval of ±20.6%.

With this reservation, Fig 8 shows mean measurements of Ta_c for z of 55, 115, 195, 495, 595 and 1195 mm over the range $135 \leq Re \leq 750$, corresponding to maximum l of 0.41. Near-insensitivity of Ta_c to Re occurs only for $z = 1195$ mm but in the two ranges $135 \leq Re \leq 300$ and $400 \leq Re \leq 750$. Though for smaller N of 0.5, the lower Ta_c measurements of Takeuchi and Jankowski using suspended aluminum-flake flow visualisation in a test section of minimum $l = 0.71$ imply that even for $z = 1195$ mm the tangential flow is underdeveloped, and the locus for $l = 0.2$ confirms this for smaller z, where Ta_c always increases with Re. While there is some scatter at large l, its relation with Ta_c in Fig 9 suggests that Re influences Ta_c essentially through l in both developing and developed tangential flow, the empirical relation for the former being:

$$Ta_c = \frac{9845}{l^{0.824}} \tag{5}$$

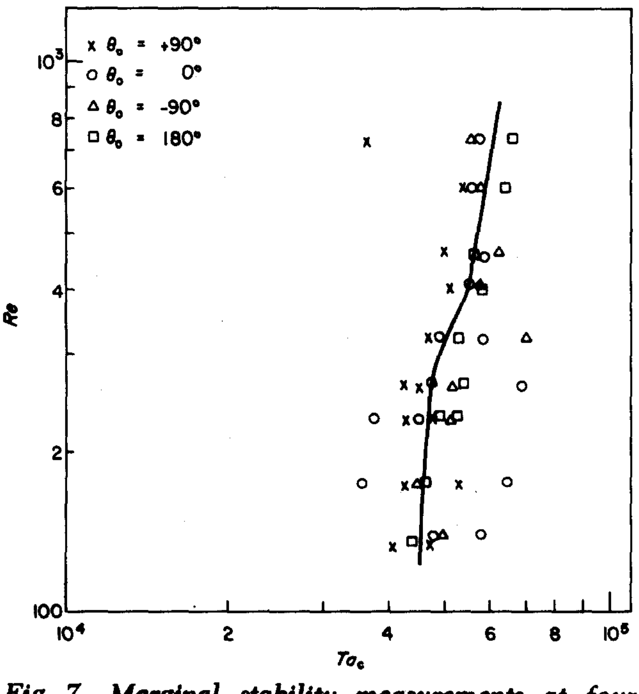

Fig 7 Marginal stability measurements at four azimuthal locations around the annulus of N = *0.565 at* z = *1195 mm*

Despite the discrepancy reported by Takeuchi and Jankowski, and shown in Fig 8, between their predictions and measurements for developed tangential flow, both indicate a reduction in Ta_c, following its maximum value, with further increase in Re. Except for $N = 0.809$ in Fig 5, this is not apparent in our measurements. Though these cover N between 0.909 and 0.565, but with a large variation in gap width, it may be argued that many other measurements in the literature in this range of N are available for comparative purposes. The same criticism may be made of the restricted comparisons in Fig 2, where the predictions of Hasoon and Martin[10] and Di Prima and Pridor[11] are shown only in relation to our measurements for $N = 0.909$. The next section is, therefore, devoted to a more general comparison of neutral stability predictions and measurements.

General comparisons

As will be seen below, consideration of measurements and predictions relating Ta_c to Re for ostensibly fully-developed tangential flow suggests four fairly well-defined regions of neutral stability:

(a) In the region $0 < Re < 80$, Ta_c increases only marginally with Re, with negligible dependence on flow conditions. Earlier well-established findings, however, show that Ta_c does vary inversely with N, increasing from 3416 for $N \to 1.0$ to 6040 for $N = 0.5$.

(b) For some range of $Re > 80$, Ta_c increases monotonically with Re according to a power-law relationship whose constant and exponent may vary considerably; theoretical research suggests that their values depend on the flow assumptions implicit in the predictions as well as N.

(c) Over a range of larger Re, the dependence of Ta_c on Re weakens considerably until in some cases not only is complete independence achieved, but the relation may become inverse, so that Ta_c diminishes as Re increases. This less stable region has recently been predicted, but only for $N = 0.5$, as indicated above, and may also depend on other geometrical factors.

(d) At still larger Re, the direct dependence of Ta_c on Re may be re-established, though not in accordance with the type of power-law relationship in (b). To what extent this is influenced by the imminence of the transition to turbulence is unknown, although in a number of cases it is undoubtedly the consequence of a reversion to developing tangential flow.

Not all measurements show clear evidence of regions (b), (c) and (d), in fact much the contrary. Thus the Re range for any of the three may be vanishingly small, significant or large, although (c) is a prerequisite for (d). These points are illustrated in Fig 10 where the measurements shown are representative of others which have been omitted for clarity; all are detailed in Table 2 and those presented in Fig. 10 are identified by reference number. Neglecting at this stage any influence of N, they fall into three groups, also indicated in Table 2. The first group is represented by the measurements of Sorour and Coney[18] for $N = 0.8$, where region (b) of power-law dependency is very restricted and early instability occurs, with low Ta_c for given Re, corresponding to region (c). In the second group, region (b) is maintained to Re around 300, as indicated by our measurements for N of 0.909 and 0.809. Over the higher Re range of region (c), Ta_c approaches or achieves independence of Re and beyond the maximum becomes inversely dependent on it; the measurements of Kaye and Elgar[21] for $N = 0.82$ show the further extension to region (d). As depicted by the results of Yamada[31] for $N = 0.895$, the third group is one of greater flow stability, where region (b) is more extensive, with region (c) never well established and no evidence of region (d).

The variation in Ta_c is, therefore, extraordinarily wide, with measurements differing by a factor of as much as 3.4 at $Re = 1000$; none is adequately forecast for $Re > 300$ by the predictions in Fig 10 of Hasoon and Martin[10], Di Prima and Pridor[11] and Chung and Astill[12], although all are for $N = 0.9$, or by those of Hughes and Reid[32] for a narrow gap. Although covering only Re up to 300, the predictions

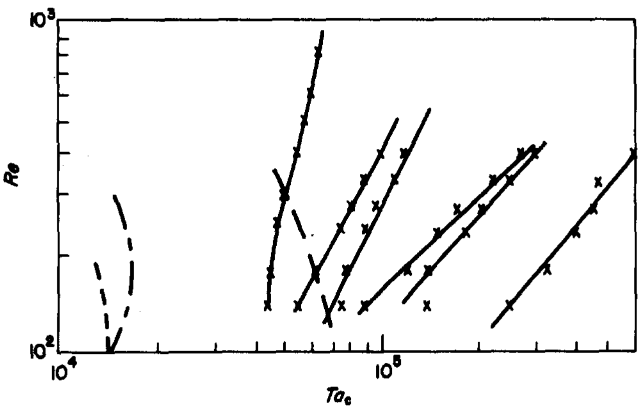

Fig 8 Mean marginal stability measurements in annulus of N = *0.565 for z (from right to left) of 55, 115, 195, 495, 595 and 1195 mm; ——, locus of l = 0.2; ———, measurements and – – –, predictions of Takeuchi and Jankowski[1] for* N = *0.5*

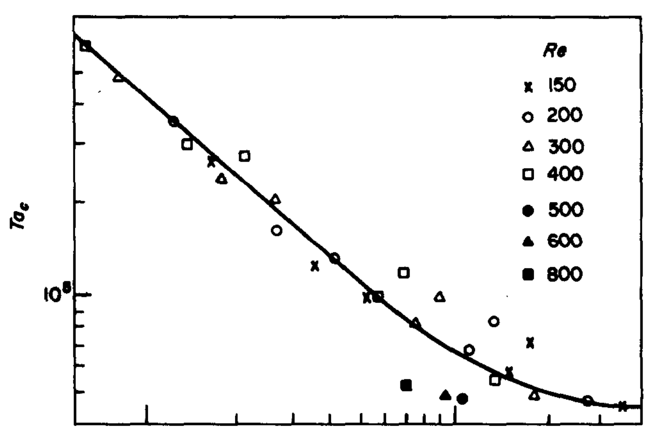

Fig 9 Mean marginal stability measurements as a function of the dimensionless axial co-ordinate l for various Re *in annulus of* N = *0.565*

of Takeuchi and Jankowski[1] at least accord qualitatively with their own measurements in Fig 10 for $N = 0.5$; being in the first group, both display the characteristic features of region (c) resulting from the destabilising influence of non-axisymmetric spiral vortices. Other measurements in the first group suggest that this is true for larger N, but the question arises as to the reasons for the two other groups, where the flow is stabilised by a monotonic power law identified with region (b) to larger Re.

As indicated above, Sorour and Coney[18] argued that, at least for a narrow gap, stability might be enhanced by sufficiently high values of either the geometrical factor S' or the end effects parameter S defined by Eqs (1) and (2). Accordingly Table 2 includes calculated values of S' and S for Re of 200, 700 and 1000, based on the quoted values of N, for all investigations cited. The inverse relation between N and Ta_c in region (a) suggests that N might independently have some influence on stability, as will be seen below. As noted earlier, the other factor promoting flow stability is the maintenance or restoration of developing tangential flow if l is less, or becomes less at high Re, than the minimum for developed tangential flow. The inverse power-law relating Ta_c and l of Eqs (3), (4) and (5) then applies. Table 2 therefore includes for each investigation the Re value for $l = 0.2$, the maximum Re for which measurements were made, and the corresponding minimum l achieved.

The correlation in Table 2 between increasing S' (as a measure of stability) and increasing group number appears satisfactory for a majority of the investigations, but not for those marked by dashes in the second and third groups, the latter reducing the average S' to below that for the second group. The same is true of S at $Re = 200$, but for Re of 700 and 1000 the correlation based on the average values of S is wholly consistent and lends weight to its validity as a stability predictor under these circumstances. If N is a measure of stability, its average values imply stability increasing with N, which is the reverse of that for region (a) at low Re. Ng and Turner[33] have recently predicted such reversal to occur at $Re \simeq 120$, at which value Ta_c is presumably insensitive to N. It is noteworthy that the measurements in the three groups cover much the same range of N, the extremes being 0.5 and 0.977.

The presence in the second group of the measurements of Flower *et al*[34] for 0.703, and in the third group those of Gravas and Martin[20] for $N = 0.576$, rather than the first group, despite their markedly below-average values of N, S' and S, may well be attributable to reversion to developing tangential flow; in both cases minimum $l < 0.2$. The same may be argued for the absence from the first group of the low S' and S values of Flower *et al*[34] for $N = 0.864$, Hasoon and Martin[10] for $N = 0.9$ and Yamada[31] for N of 0.895 and 0.955, though for higher-than-average N. This parameter also appears sufficiently large and influential to ensure second and third group stability respectively for the measurements of Yamada[31] for N of 0.971 and 0.931 even when, as in these cases, $l > 0.2$ and values of S' and S are well below average. However, N is not equally effective in stabilising the first-group measurements

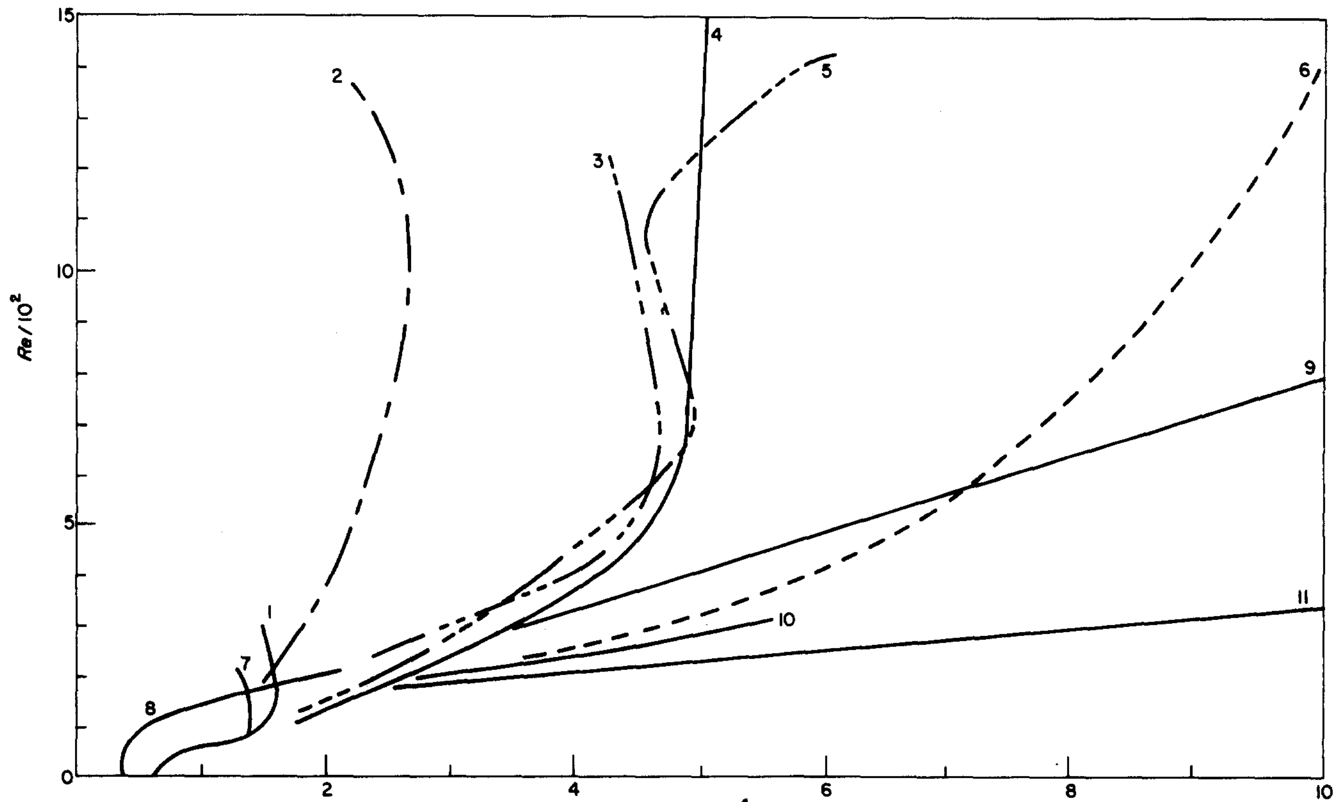

***Fig 10** Comparison of present and previous marginal stability measurements and theoretical predictions. See Table 2 for key to reference numbers*

of Cornish[17] for $N = 0.977$, with comparably-low values of S' and S and the largest N of all, although Cole[35] has questioned Cornish's use of pressure-drop measurements to detect the onset of instability, and Snyder[13] regards them as erroneous.

This relates to the point in the introduction that discrepancies in measured marginal stability in Fig 10 may arise in part from the sensitivity of the apparent onset of instability to the disturbance level or to the method of observation. The wide variety of experimental work considered unfortunately precludes estimates of standard deviations, error analysis or comparison of uncertainty intervals for the different methods of observation, which include flow visualisation techniques, hot-wire anemometry and pressure-drop measurements. The necessary information is not available in many of the papers cited. It is nevertheless scarcely conceivable that sensitivity errors could account for discrepancies in Ta_c measurements of 100% between the first and second groups, let alone over 300% between the first and third groups. If Takeuchi and Jankowski's predictions for $N = 0.5$ are correct, their larger Ta_c measurements represent at worst 28% error over the Re range covered.

Perhaps the most consistent feature in the foregoing is the presence of developing tangential flow at low l; particularly striking is the reversion to region (d) following region (c) in the second-group measurements in Fig 10 of Kaye and Elgar[21] for $N = 0.82$. Fig 11 incorporates present empirical correlations between Ta_c and l for developing tangential flow, described by Eqs (3), (4) and (5) for N of 0.909, 0.809 and 0.565 respectively, together with the earlier correlations of Martin and Payne[16] for $N = 0.727$, Payne and Martin[23] and Martin and Hasoon[24], both for $N = 0.9$. These are represented respectively by:

$$Ta_c = \frac{2300}{l^{1.175}}; \; 200 < Re < 1700; \; 0.01 < l < 0.15 \quad (6)$$

$$Ta_c = \frac{1554}{l^{1.23}}; \; 365 < Re < 1510; \; 0.006 < l < 0.04 \quad (7)$$

$$Ta_c = \frac{5523}{l^{0.93}}; \; 200 < Re < 400; \; 0.008 < l < 0.03 \quad (8)$$

Table 2 Summary of flow and geometrical factors for the present and previous investigations

MEASUREMENTS	Ref. in Fig 10	Group No.	N	S'	S at $Re=200$	S at $Re=700$	S at $Re=1000$	Re for $l=0.2$	Re_{max}	l_{min}
Takeuchi and Jankowski[1]	1	1	0.5	3.14	2.05	—	—	1000	280	0.714
Sorour and Coney[18]	2	1	0.8	3.26	2.08	0.724	0.531	1290	1160	0.222
Astill[28]		1	0.727	2.71	2.50	0.649	0.446	706	1500	0.094
Cornish[17]		1	0.977	2.18	1.26	0.453	0.362	6000	1190	1.008
Kaye and Elgar[21]		1	0.735	3.33	2.26	0.719	0.552	700	1480	0.095
Average Values			0.748	2.92	2.03	0.636	0.473			
Donnelly and Fultz[36]		2	0.95	3.43	2.66	—	—	2848	200	2.848
Grosvenor[37]		2	0.9	4.94	4.63	1.45	—	1150	860	0.267
This study	3	2	0.809	2.15	1.77	0.654	0.466	810	2100	0.077
This study	4	2	0.909	3.53	2.73	1.09	0.780	2080	2100	0.198
Williamson[19]		2	0.9	6.82	5.23	2.16	1.52	3880	1780	0.436
Kaye and Elgar[21]	5	2	0.82	4.39	3.41	1.40	0.947	1150	1220	0.189
Becker and Kaye[22]		2	0.81	2.99	2.43	0.954	0.701	1095	1650	0.133
Flower *et al*[34]		2	0.703	2.73	—	0.844	0.631	870	1250	0.139
Flower *et al*[34]		2′	0.864	1.57	1.09	0.499	0.361	775	1250	0.124
Yamada[31]		2′	0.971	1.76	1.52	—	—	1950	420	0.929
This study		2′	0.565	1.13	1.22	0.395	—	275	750	0.073
Average Values			0.836	3.22	2.67	1.05	0.772			
Gravas and Martin[20]		3	0.81	4.04	3.27	1.58	1.20	1247	1520	0.164
Gravas and Martin[20]		3	0.9	4.94	4.21	1.96	1.49	2188	1600	0.274
Sorour and Coney[18]		3	0.955	6.58	4.77	2.48	2.06	5710	1390	0.822
Yamada[31]	6	3′	0.895	0.95	0.84	0.379	0.283	540	1400	0.077
Hasoon and Martin[10]		3′	0.9	1.62	1.20	—	—	460	410	0.224
Yamada[31]		3′	0.931	1.15	0.83	—	—	810	525	0.309
Yamada[31]		3′	0.955	1.41	1.18	0.634	0.490	1240	1420	0.173
Gravas and Martin[20]		3′	0.576	2.06	1.85	—	—	398	510	0.156
Average values			0.865	2.84	2.27	1.41	1.10			
PREDICTIONS										
Takeuchi and Jankowski[1]	7	1	0.5							
Di Prima and Pridor[11]	8	3	0.9							
Hasoon and Martin[10]	9	3	0.9							
Hasoon and Martin[10]		3	0.5							
Chung and Astill[12]	10	3	0.9							
Chung and Astill[12]		3	0.576							
Hughes and Reid[32]	11	3	→1.0							

Eqs (7) and (8) are wholly empirical, but Eq (6) results from combining finite-difference predictions[16] of tangential boundary-layer displacement thickness δ_θ as a function of l for $0.05 < N < 0.98$ with Astill's[26] empirical stability criterion for $N = 0.727$, given by:

$$\frac{\omega_c^2 R_1 \delta_\theta^3}{\nu^2} \geqslant 576 \tag{9}$$

Also shown in Fig 11 are the predictions of Martin and Hasoon[24] for N of 0.5 and 0.9 based on a normal-mode analysis of the linearised disturbance equations and a quasi-fully developed approch, where at a given l the tangential velocity profile is treated as developed and formulated as a characteristic-value problem. Although the analysis covered the range $50 \leqslant Re \leqslant 400$, the relative errors increased with Re as the large difference matrices approached the magnitude of the developed matrices for the radial and tangential velocity disturbances. For this reason, and in the interests of clarity, only predictions for $Re = 300$ are included in Fig 11.

While these predictions are mostly in the measured Ta_c range covered by Eqs (3) to (8), and confirm an inverse dependence of Ta_c on l, they are not of power-law type. They nevertheless support the available empirical evidence that a decrease in N stabilises developing tangential flow to some extent, but this diminishes with l, virtually to vanishing point in the region of $l = 0.01$. At lower l the influence of N on stability may even be reversed.

The annuli on which Eqs (3) to (8) are based are too few to assess the effects on stability of the geometrical factor S' and hence the end effects parameter S. But despite the relatively limited attention

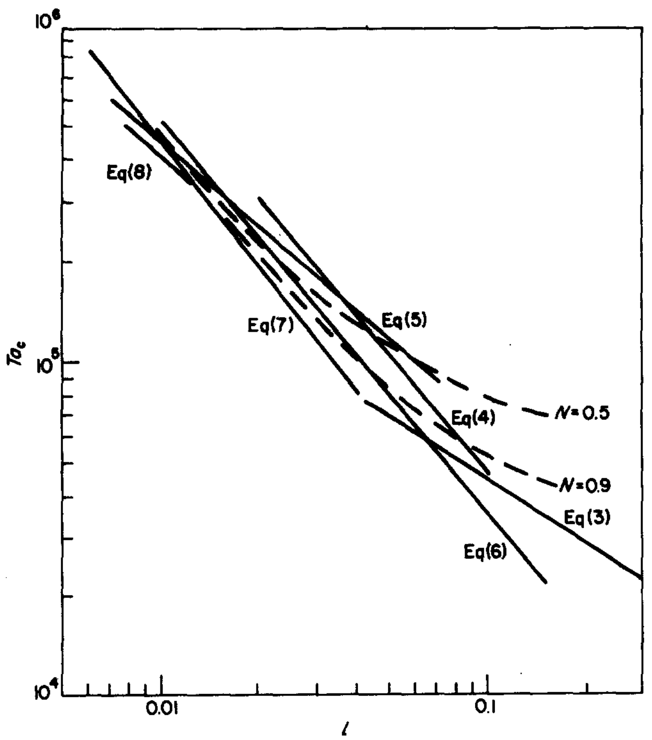

Fig 11 Comparison of present and previous marginal stability measurements and predictions in developing tangential flow; ---, predictions of Martin and Hasoon[24] for Re = *300*

given to marginal stability in developing tangential flow, the predictions and correlations in Fig 11 display much greater consistency than those for developed tangential flow in Fig 10. Nowhere in the range $0.01 < l < 0.1$ do Eqs (3) to (8) indicate values of Ta_c which, for given l, differ by a factor greater than about 1.7; for the predictions the factor is less than 1.5, notwithstanding the variation in N.

The authors believe that the destabilising effect of non-axisymmetric spiral vortices reported by Takeuchi and Jankowski is a notable advance and from a theoretical standpoint should be extended to N between 0.5 and unity and higher Re; non-linear analyses may be required if precision is to be improved. The foregoing survey of a large number of disparate stability measurements for $0.5 \leqslant N \leqslant 0.977$ also indicates the existence of countervailing influences capable of stabilising the flow. For developed tangential flow these appear to include relatively high values of N, S' and S. The flow may be further stabilised by values of l below that for developed tangential flow in which there is an initial instability.

The precise effects of these, and any other stabilising parameter yet to be identified, would be best determined by systematic experimental research based on existing knowledge and pursued in conjunction with the above theoretical programme. This should encompass the sensitivity of the entrance length (in terms of l) for developing tangential flow with an initial instability to diminishing Re. It should also accommodate studies of the vortex development length as defined by Takeuchi and Jankowski, involving the sensitivity of the apparent location of neutral stability to the method of observation, and hence comparative studies of different methods.

Conclusions

The following conclusions may be drawn:

- Over the ranges examined, the measurements made confirm earlier findings that irrespective of azimuthal orientation, radius ratio and axial-flow Reynolds number, marginal instability originates close to the inner rotating cylinder and spreads radially outwards with further increases in Taylor number.
- Within assessed uncertainty intervals, the axisymmetry of Ta_c measurements indicates that the initial disturbance is not sensitive to small manufacturing and locational variations in gap width and that the non-axisymmetry of spiral vortex flow is not reflected in orientational variations in Ta_c.
- For all three values of N investigated, Ta_c is found to increase monotonically with Re at any given z in the earlier stages of developing tangential flow; in this region Ta_c diminishes as the dimensionless axial co-ordinate $l = 2z/[Re(R_2 - R_1)]$ increases. When the flow approaches full development, Ta_c becomes insensitive to l at a value of the latter which increases markedly at low Re.
- In developed tangential flow Ta_c tends to, and may achieve, independence of Re, but at larger values of the latter than previously reported for a wide gap. This effective destabilisation is currently

ascribed to the effect of increasing azimuthal wavenumber of the spiral vortex which then forms the initial instability.

- Comparison of present measurements and others available in the literature suggests that in developed tangential flow the above destabilising influence may be offset at larger Re by sufficiently high values of the geometric factor S', the end effects parameter S and the radius ratio N. This is a reversal of the latter's effect at low Re.
- The flow may also be stabilised if l becomes sufficiently small at high Re to cause a reversion to developing tangential flow, when the initial instability ceases to be of spiral vortex form.
- As was found possible in earlier stability measurements for developing tangential flow where, as noted above, Ta_c varies inversely with l, these observations are correlated by power-law relations between those parameters. In the range $0.01 < l < 0.1$ agreement with previous correlations is satisfactory and when taken together they indicate that, as predicted, stability is improved by a reduction in N. Such improvement is most marked at large l; for values around 0.01 measurements again confirm predictions that Ta_c is little influenced by N.

Acknowledgements

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